WAIST OF THE SPHERE FOR MAPS TO MANIFOLDS

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Abstract. We generalize the sphere waist theorem of Gromov and the Borsuk–Ulam type measure partition lemma of Gromov–Memarian for maps to manifolds.

1. Introduction

In [4, 12] the sphere waist theorem was proved for a continuous map from a sphere S^n to the Euclidean space $\mathbb{R}^m$, showing that the preimage of some point is "large enough". Here we generalize it for maps from the sphere to any m-dimensional manifold.

Let the sphere S^n be the standard unit sphere in $\mathbb{R}^{n+1}$. Denote the standard probabilistic measure on S^n by μ, denote by $U_\varepsilon(X)$ the ε-neighborhood of $X \subseteq S^n$ with respect to the standard metric on S^n.

Theorem 1.1. *Suppose $h\colon S^n \to M$ is a continuous map from the n-sphere to m-manifold with $m \le n$. In case $m = n$ let the homology map $h_*\colon H_n(S^n, \mathbb{F}_2) \to H_n(M, \mathbb{F}_2)$ be trivial. Then there exists a point $z \in M$ such that for any $\varepsilon > 0$*

$$\mu U_\varepsilon(h^{-1}(z)) \ge \mu U_\varepsilon S^{n-m}$$

Here S^{n-m} is the $(n-m)$-dimensional equatorial subsphere of S^n, i.e. $S^{n-m} = S^n \cap \mathbb{R}^{n-m+1}$.

In this paper we extend the topological reasoning to the case of maps to manifolds; for the geometrical and analytical part of the proof the reader is referred to [12].

2. The corresponding generalization of the Borsuk–Ulam theorem

Following [12], we are going to prove the corresponding analogue of the Borsuk–Ulam theorem first. In fact, we will prove a more general Borsuk–Ulam type theorem, following mostly [8].

Let us give some definitions. Consider a compact topological space X with a probabilistic Borel measure μ. Let $C(X)$ denote the set of continuous functions on X.

Definition 2.1. A finite-dimensional linear subspace $L \subset C(X)$ is called *measure separating*, if for any $f \ne g \in L$ the measure of the set

$$e(f, g) = \{x \in X : f(x) = g(x)\}$$

is zero.

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In particular, if X is a compact subset of $\mathbb{R}^n$ (or S^n) such that $X = \mathrm{cl}(\mathrm{int}\, X)$, μ is any absolutely continuous measure, then any finite-dimensional space of analytic functions is measure-separating, because the sets $e(f, g)$ always have dimension $< n$ and therefore measure zero. Then for any collection of q elements of a measure-separating subspace we define a partition of X.

Definition 2.2. Suppose $F = \{u_1, \dots, u_q\} \subset C(X)$ is a family of functions such that $\mu(e(u_i, u_j)) = 0$ for all $i \neq j$. The sets (some of them may be empty)

$$V_i = \{x \in X : \forall j \neq i\ u_i(x) \geq u_j(x)\}$$

have a zero measure overlap, so they define a partition $P(F)$ of X. In case u_i are linear functions on $\mathbb{R}^n$ we call $P(F)$ a *generalized Voronoi partition*.

Note that if we consider the standard sphere $S^n \subseteq \mathbb{R}^{n+1}$, and homogeneous linear functions $F \subset C(\mathbb{R}^{n+1}) \subset C(S^n)$, then $P(F)$ is always a partition into convex subsets of S^n, or a partition consisting of one set equal to the whole S^n. The same is true for (non-homogeneous) linear functions on the Euclidean space $\mathbb{R}^n$.

We have to generalize the notion of a center function from [12].

Definition 2.3. Let $L \subset C(X)$ be a finite-dimensional linear subspace of functions. Suppose that for any subset $F \subset L$ such that all sets $\{V_1, \dots, V_q\} = P(F)$ have nonempty interiors we can assign *centers* $c(V_1), \dots, c(V_q) \in X$ to the sets. If this assignment is continuous w.r.t. F and equivariant w.r.t. the permutations of functions in F and permutations of points in the sequence $c_1, \dots, c_q$, we call $c(\cdot)$ a *q-admissible center function* for L.

Now we are ready to state the generalization of [12, Theorem 3].

Theorem 2.4. *Suppose L is a measure-separating subspace of $C(X)$ of dimension $n+1$, $\mu_1, \dots, \mu_{n-m}$ ($n > m$) are absolutely continuous (w.r.t. the original measure on X) probabilistic measures on X. Let $q = p^\alpha$ be a prime power, $c(\cdot)$ be a q-admissible center function for L, and*

$$h : X \to M$$

be a continuous map to an m-dimensional manifold. Suppose also that the cohomology map $h^*\colon H^i(M, \mathbb{F}_p) \to H^i(X, \mathbb{F}_p)$ is a trivial map for $i > 0$.

Then there exists a q-element subset $F \subset L$ such that for every $i = 1, \dots, n-k$ the partition $P(F)$ partitions the measure μ_i into q equal parts, and we also have

$$h(c(V_1)) = h(c(V_2)) = \dots = h(c(V_q))$$

for $\{V_1, \dots, V_q\} = P(F)$.

Remark 2.5. It is clear from the proof in Section 4 that instead of μ_1 we can take any "charge" (i.e. a measure that can be negative), the only essential requirement is that $\mu_1(X) \neq 0$. This requirement guarantees that all the partition sets $V_1, \dots, V_q$ have nonempty interiors.

The other measures $\mu_2, \dots, \mu_{n-k}$ may be replaced by arbitrary functions of the parts $V_1, \dots, V_q$ that depend continuously on the partition under the assumption that all the interiors of V_j ($j = 1, \dots, q$) are nonempty.

When $X = \mathbb{R}^{n+1}, S^n$ and L is a set of linear functions (this is the case needed in the proof of Theorem 1.1) the partition sets are convex and there are a lot of suitable functions, for example the Steiner measures (compare [8]). The existence of many admissible center functions is also obvious in this case.

3. Topological facts

In this section we remind some facts from equivariant topology and prove several Borsuk–Ulam–Bourgin–Yang type results that are needed in the proof of Theorem 2.4.

3.1. **Representations and transfer.** Let us start from the following typical problem: let G be a finite group, Y be a G-space (i.e. a topological space with a continuous action of G) and V be a finite-dimensional linear G-representation. For any continuous G-equivariant map $f\colon Y \to V$ we may guarantee that $f^{-1}(0)$ is non-empty if the G-equivariant Euler class of the vector bundle $Y \times V$ is nonzero. Indeed, the map f can be naturally considered as a G-equivariant section of this bundle. This Euler class is the natural image of the "universal" Euler class

$$e(V) \in H_G^{\dim V}(\mathrm{pt}, \mathbb{Z}_V) = H^{\dim V}(BG, \mathcal{O}).$$

In this formula pt is a one-point space with trivial G-action, $\mathbb{Z}_V$ is the group $\mathbb{Z}$ considered to have the G-action same as the determinant of its action on V, and $\mathcal{O}$ denotes the corresponding to $\mathbb{Z}_V$ quotient sheaf over BG.

When we consider the Euler class $e(V)$ over Y we assume it to be contained in the cohomology $H_G^{\dim V}(Y, \mathbb{Z}_V)$.

To avoid the twisted cohomology in our particular problem we are going to use the following consideration. Let F be a subgroup of G. Then in the above situation we have two Euler classes

$$e_G(V) \in H_G^{\dim V}(Y, \mathbb{Z}_V), \quad e_F(V) \in H_F^{\dim V}(Y, \mathbb{Z}_V),$$

where $\mathbb{Z}_V$ is simultaneously a G-module and an F-module (see [14]). If F acts on V with positive determinant then the last class resides in $H_F^{\dim V}(Y, \mathbb{Z})$.

There exists a natural map $\pi^*\colon H_G^*(Y, \mathbb{Z}_V) \to H_F^*(Y, \mathbb{Z}_V)$ that takes $e(V)$ in G-equivariant cohomology to $e(V)$ in F-equivariant cohomology. There also exist the transfer homomorphism [2]

$$\pi_!\colon H_F^*(Y, \mathbb{Z}_V) \to H_G^*(Y, \mathbb{Z}_V)$$

such that the composition $\pi_! \circ \pi^*$ is a multiplication by $|G/F|$. If we tensor-multiply $\mathbb{Z}_V$ by $\mathbb{F}_p$, and if p is not a divisor of $|G/F|$ then $\pi^* : H_G^*(Y, \mathbb{Z}_V \otimes \mathbb{F}_p) \to H_F^*(Y, \mathbb{Z}_V \otimes \mathbb{F}_p)$ is a monomorphism. If F acts on V with positive determinant then the last group is $H_F^*(Y, \mathbb{F}_p)$ without any twist.

3.2. **The permutation group and its Sylow subgroup.** Now we take $G = \Sigma_q$, the permutation (symmetric) group, where $q = p^k$ is a prime power and $\Sigma_q^{(p)}$ is its p-Sylow subgroup (for example if $q = p$ is a prime, p-Sylow subgroup of Σ_p is a cyclic group of prime order p). The group Σ_q acts on $\mathbb{R}^q$ by permutations of coordinates. Since the diagonal $\Delta \subset \mathbb{R}^q$ is a Σ_q-invariant subspace, Σ_q acts on the quotient $\mathbb{R}^q/\Delta$, which is isomorphic to $\mathbb{R}^{q-1}$. We denote this representation

$$\alpha_q = \{(x_1, \dots, x_q) \in \mathbb{R}^q : x_1 + \dots + x_q = 0\}.$$

Definition 3.1. Denote the Euler class of α_q reduced modulo p by

$$\theta \in H^{q-1}(B\Sigma_q, \mathbb{Z}_{\alpha_q} \otimes \mathbb{F}_p).$$

Note that the m-th power of θ resides in the cohomology with coefficients $\mathbb{Z}_{\alpha_q} \otimes \mathbb{F}_p$ if m is odd and $\mathbb{F}_p$ is m is even.

Definition 3.2. Denote by $\theta_p \in H^{q-1}(B\Sigma_q^{(p)}, \mathbb{F}_p)$ the image of θ under the natural map $\pi^* : H^*(B\Sigma_q, \mathbb{Z}_{\alpha_q} \otimes \mathbb{F}_p) \to H^*(B\Sigma_q^{(p)}, \mathbb{F}_p)$. Note that θ_p resides in non-twisted cohomology, because $\mathbb{Z}_{\alpha_q} \otimes \mathbb{F}_2 = \mathbb{F}_2$ for $p = 2$ and $\Sigma_q^{(p)}$ preserves the orientation for odd p.

It is well-known [16, 10] that the powers of θ_p are all non-trivial in $H^*(B\Sigma_q^{(p)}, \mathbb{F}_p)$, it may be shown by passing to the elementary Abelian p-torus $(\mathbb{Z}_p)^k \subseteq \Sigma_q^{(p)}$ (note that $q = p^k$).

Since $\Sigma_q^{(p)}$ is a Sylow subgroup of Σ_q then the index $|\Sigma_q/\Sigma_q^{(p)}|$ is not divisible by p and the power θ^m is nonzero over a Σ_q-space Y iff the power θ_p^m is nonzero over Y.

3.3. **Configuration spaces.** Remind the definition of the configuration space:

Definition 3.3. Denote by $K^q(\mathbb{R}^d) \subset (\mathbb{R}^d)^q$ the space of all ordered q-tuples of distinct elements in $\mathbb{R}^d$, i.e. the *configuration space* of $\mathbb{R}^d$. This space has the natural Σ_q-action and $\Sigma_q^{(p)}$-action.

Denote the natural image of θ_p in the equivariant cohomology of any $\Sigma_q^{(p)}$-space by the same letter θ_p, it does not lead to a confusion in this paper. We give a variant of [7, Lemma 6] (the essential idea is from [15]).

Lemma 3.4.

$$\theta_p^{d-1} \neq 0 \in H^*_{\Sigma_q^{(p)}}(K^q(\mathbb{R}^d), \mathbb{F}_p).$$

Proof. This is shown for θ^{d-1} in [7], and this result follows from the above reasoning with transfer. ☐

We also need the following lemma:

Lemma 3.5. *Suppose θ_p^m is nonzero over a $\Sigma_q^{(p)}$-space Y. For a $\Sigma_q^{(p)}$-equivariant map*

$$f\colon Y \to \alpha_q$$

denote

$$Z_f = f^{-1}(0).$$

Then θ_p^{m-1} is nonzero over Z_f.

Proof. Since the map $Y \setminus Z_f \to \alpha_q$ has no zeroes, the class θ_p is zero over $Y \setminus Z_f$. If we assume $\theta_p^{m-1}|_{Z_f} = 0$ we would obtain (it is important here to use the Čech cohomology) $\theta_p^m = 0$ over the entire Y. ☐

3.4. **Remarks on the notation.** Lemma 3.5 guarantees that some $\Sigma_q^{(p)}$-orbit of Y is mapped to one point under any continuous map $Y \to \mathbb{R}^m$, similar to the standard Borsuk–Ulam theorem. Now we are going to extend this result for maps to manifolds. We need to fix some notation first.

The constant coefficients $\mathbb{F}_p$, where $\mathbb{F}_p$ is the field with p elements, are suppressed in the notation of ordinary and equivariant cohomology groups. Any other coefficients are always indicated.

Let X be a G-space, $A \subset X$ be an invariant subspace and $\alpha \in H^*_G(X)$. In what follows by $\alpha|_A \in H^*_G(A)$ we denote the image of α under the homomorphism induced by the inclusion $\iota_A\colon A \subset X$, and by $\eta|_X$ the image of $\eta \in H^*_G(\mathrm{pt})$ in $H^*_G(X)$ under the homomorphism of the equivariant cohomology induced by the map $X \to \mathrm{pt}$.

3.5. **The Haefliger class.** Denote by T a p-torus group such that $|T| = q$, so $q = p^\alpha$. Consider an embedding of T in Σ_q via regular representation (the embedding is not unique, it depends on the ordering of elements of T). Assume that G is a p-subgroup of Σ_q containing T and $\Sigma_q^{(p)}$ is a p-Sylow subgroup of Σ_q containing G. Thus we have $T \subseteq G \subseteq \Sigma_q^{(p)}$. In main results of this paper $G = \Sigma_q^{(p)}$.

We saw that the Euler class

$$\theta = \theta_{\Sigma_q} := e_{\Sigma_q}(\alpha_q) \in H^{q-1}(B\Sigma_q; \mathcal{O})$$

is mapped to $\theta_p \in H^{q-1}_{\Sigma_q^{(p)}}(\mathrm{pt})$. The inclusions $T \subseteq G \subseteq \Sigma_q^{(p)}$ induce the homomorphisms

$$H^{q-1}_{\Sigma_q^{(p)}}(\mathrm{pt}) \to H^{q-1}_G(\mathrm{pt}) \to H^{q-1}_T(\mathrm{pt}),$$

under which we have $\theta_p \to \theta_G \to \theta_T$.

Following [5] we are going to define the equivariant diagonal class $\gamma_{M,G} \in H^{m(q-1)}_G(M^q)$ possessing the properties given in the following lemma (we denote by M^q the q-th Cartesian power of M):

Lemma 3.6. *Let M be an m-dimensional topological manifold. There exists a class $\gamma_{M,G} \in H^{m(q-1)}_G(M^q)$ such that:*

1) For any point $y \in M$ the image of $\gamma_{M,G}$ in $H^{m(q-1)}_G(y^q) = H^{m(q-1)}_G(\mathrm{pt})$ coincides with θ^m_G.

2) The image of $\gamma_{M,G}$ in $H^{m(q-1)}_G(M^q \setminus \Delta)$ is trivial.

3) For an open submanifold $U \subset M$ the image of $\gamma_{M,G}$ under the homomorphism $H^{m(q-1)}_G(M^q) \to H^{m(q-1)}_G(U^q)$ coincide with $\gamma_{U,G}$.

4) $\gamma_{M,G}$ is the image of $\gamma_{M,p} := \gamma_{M,\Sigma_q^{(p)}}$, (the Haefliger class corresponding to the Sylow subgroup $\Sigma_q^{(p)}$) under the restriction homomorphism $H^{m(q-1)}_{\Sigma_q^{(p)}}(M^q) \to H^{m(q-1)}_G(M^q)$.

Proof. It is sufficient to consider a connected manifold without boundary (if the boundary is nonempty we construct the class for its double and then restrict it to M).

Let us first give an explanation for a closed connected orientable smooth manifold M. In this case the equivariant Thom class of the normal bundle to the diagonal $\Delta \subset M^q$ can be considered as an element of the group $H^{m(q-1)}_G(U_\varepsilon(\Delta), \partial U_\varepsilon(\Delta))$ where $U_\varepsilon(\Delta)$ is a tubular ε-neighborhood of Δ in M^q where ε is small enough. This group is isomorphic to $\mathbb{F}_p$; and by the excision axiom is also isomorphic to $H^{m(q-1)}_G(M^q, M^q \setminus \Delta)$. For an open ball $U \subset M$ we can take a generator $\xi_{U,G} \in H^{m(q-1)}_G(U^q, U^q \setminus \Delta(U))$ which is mapped to θ^m_G under the homomorphism

$$\mathbb{F}_p = H^{m(q-1)}_G(U^q, U^q \setminus \Delta(U)) \to H^{m(q-1)}_G(U^q) = H^{m(q-1)}_G(\mathrm{pt}).$$

The inclusion $U \subset M$ induces an isomorphism

$$\mathbb{F}_p = H^{m(q-1)}_G(M^q, M^q \setminus \Delta(M)) \to H^{m(q-1)}_G(U^q, U^q \setminus \Delta(U))$$

and we denote by $\xi_{M,G}$ the generator that is mapped to $\xi_{U,G}$. Finally, denote by $\gamma_{M,G}$ the image of $\xi_{M,G}$ in $H^{m(q-1)}_G(M^q)$. The class $\gamma_{M,G}$ possesses all desired properties as can be easily verified.

Now consider the orientable case. Then either $p = 2$ and M is an arbitrary manifold (orientable or not), or $p > 2$ and the manifold M is orientable.

Let E_k be any orientable manifold with a free action of $\Sigma_q^{(p)}$. For example, we can take $E_k = K^q(\mathbb{R}^k)$, the configuration space. It is $(k-2)$-connected and when $k \to \infty$ the space E_k approaches $E\Sigma_q^{(p)}$, the total space of the universal bundle $E\Sigma_q^{(p)} \to B\Sigma_q^{(p)}$.

Fixing the orientations on M and E_k, we obtain oriented manifolds $\Delta \times E_k$ and $M^q \times E_k$ where $\Delta = \Delta(M)$ is the diagonal in M^q and we identify Δ with M. Since G is a p-group, the diagonal action preserves the orientations of these manifolds. Hence we have oriented manifolds

$$\Delta \times_G E_k := (\Delta \times E_k)/G = \Delta \times (E_k/G) \quad \text{and} \quad M^q \times_G E_k := (M^q \times E_k)/G$$

of dimensions $m + r$ and $mq + r$ respectively where $r = \dim E_k$.

It follows from the Poincaré–Lefschetz duality for manifolds $\Delta \times_G E_k$ and $M^q \times_G E_k$ that

$$\mathbb{F}_p = H_{m+r}(\Delta \times_G E_k) = H_{m+r}(\Delta \times_G E_k) = \\ = H^{m(q-1)}((M^q, M^q \setminus \Delta) \times_G E_k) = H^{m(q-1)}((M^q, M^q \setminus \Delta) \times_G E_k).$$

Here we consider homology with closed supports (defined via infinite cycles). If the connectivity of the manifold E_k is large enough, then

$$H_G^{m(q-1)}(M^q) = H^{m(q-1)}(M^q \times_G E_k).$$

Similarly we have $H_G^{m(q-1)}(M^q \setminus \Delta) = H^{m(q-1)}((M^q \setminus \Delta) \times_G E_k)$ and also an isomorphism for pairs

$$H_G^{m(q-1)}(M^q, M^q \setminus \Delta) = H^{m(q-1)}((M^q, M^q \setminus \Delta) \times_G E_k).$$

For a ball $U \subset M$ the class $\xi_{U,G}$ is already defined and we define $\xi_{M,G} \in H_G^{m(q-1)}(M^q, M^q \setminus \Delta)$ as a class which is mapped onto $\xi_{U,G}$. Finally we define $\gamma_{M,G}$ as the image of $\xi_{M,G}$ in $H_G^{m(q-1)}(M^q)$.

Let us consider now the general (nonorientable) case.

Denote by $\mathcal{H}_\mathbb{Z}$ and $\mathcal{H}$ the orientation sheaves of M with fibers $\mathbb{Z}$ and $\mathbb{F}_p$ respectively. We have $\mathcal{H} = \mathcal{H}_\mathbb{Z} \otimes \mathbb{F}_p$. It is easily seen that $\mathcal{H}_\mathbb{Z} \otimes \mathcal{H}_\mathbb{Z} = \mathbb{Z}$, which is the constant sheaf. Hence $\mathcal{H} \otimes \mathcal{H} = \mathbb{F}_p$, i.e. $\mathcal{H} = \mathcal{H}^{-1}$. Now $\mathcal{H}^{\widehat{\otimes} q}$ is the orientation sheaf of the manifold M^q. It is enough to consider the case q is odd (i.e. p is odd), where the orientation of $\mathbb{F}_p$ makes sense. We have

$$\mathcal{H}^{\widehat{\otimes} q} \mid_\Delta = \mathcal{H}^{\otimes q} = \mathcal{H} \otimes (\mathcal{H} \otimes \mathcal{H})^{\otimes \frac{q-1}{2}} = \mathcal{H} \otimes \mathbb{F}_p^{\otimes \frac{q-1}{2}} = \mathcal{H}$$

Denote by $\mathcal{H}'$ and $\widetilde{\mathcal{H}}'$ the orientation sheaves of manifolds $\Delta \times E_k$ and $M^q \times E_k$ respectively. As above we have $\mathcal{H}' = \widetilde{\mathcal{H}}'|_{\Delta \times E_k}$ and $\widetilde{\mathcal{H}}' \otimes \widetilde{\mathcal{H}}' = \mathbb{F}_p$.

Let us denote the orientation sheaf for the manifold $\Delta \times_G E_k$ again by $\mathcal{H}$ and the orientation sheaf for $M^q \times_G E_k$ by $\widetilde{\mathcal{H}}$. Let us show that $\widetilde{\mathcal{H}}|_{\Delta \times_G E_k} = \mathcal{H}$ and $\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}} = \mathbb{F}_p$ (the constant sheaf on $M^q \times_G E_k$). Consider the action of the group G on the total spaces of sheaves $\mathcal{H}'$, $\widetilde{\mathcal{H}}'$ and $\widetilde{\mathcal{H}}' \otimes \widetilde{\mathcal{H}}'$. The quotient spaces are the total spaces of sheaves $\mathcal{H}$, $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}$ respectively. Now the first assertion follows easily. To show that the sheaf $\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}$ on $M^q \times_G E_k$ is the constant sheaf $\mathbb{F}_p$ we consider a nonzero global section s of the constant sheaf $\widetilde{\mathcal{H}}' \otimes \widetilde{\mathcal{H}}' = \mathbb{F}_p$ and an element $g \in \Sigma_q^{(p)}$. The image of s under g is a section αs where $\alpha = \alpha(g) \in \mathbb{F}_p$. The order of the element g is a power of p, say p^r, so g^{p^r} is the identity of the group G. Hence $\alpha^{p^r} = 1$ and from Fermat's Little Theorem we obtain that $\alpha = 1$. Therefore any section s is mapped to itself by all elements of G and thus defines a section of $\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}$, which is nowhere zero (if s is nonzero); hence $\widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}$ is constant (with fiber $\mathbb{F}_p$). It follows from the Poincaré–Lefschetz duality for manifolds $\Delta \times_G E_k$ and $M^q \times_G E_k$ that

$$\mathbb{F}_p = H_{m+r}(\Delta \times_G E_k; \mathcal{H}) = H_{m+r}(\Delta \times_G E_k; \widetilde{\mathcal{H}}) = \\ = H^{m(q-1)}((M^q, M^q \setminus \Delta) \times_G E_k; \widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}) = \\ = H^{m(q-1)}((M^q, M^q \setminus \Delta) \times_G E_k) = H_G^{m(q-1)}(M^q, M^q \setminus \Delta).$$

Here $r = \dim E_k$ and we consider homology with closed supports (defined via infinite cycles) and assume that the connectivity of the manifold E_k is large enough.

Hence there exists a nonzero class $\gamma_{M,G} \in H_G^{m(q-1)}(M^q)$ with trivial image in $H_G^{m(q-1)}(M^q \setminus \Delta)$. Similarly for an open $U \subset M$ we have a class $\gamma_{U,G} \in H_{\Sigma_q^{(p)}}^{m(q-1)}(U^q; \mathbb{F}_p)$; and we can assume that in the equivariant cohomology the restriction of $\gamma_{M,G}$ onto U^q coincides with

$\gamma_{U,G}$. Note that U is orientable (i.e. the orientation sheaf is constant) if U is small enough. If $U \subset M$ is a ball then $\gamma_{U,G} \in H_G^{m(q-1)}(U^q) = H_G^{m(q-1)}(\mathrm{pt})$ coincides up to a constant nonzero factor with θ_G^m since the cohomology $H_G^{m(q-1)}(U^q, U^q \setminus \Delta(U))$ is generated by the equivariant Thom class of the normal bundle of $\Delta(U)$ in U^q.

Let us prove claim 4 of the theorem. Since $\theta_T^k \neq 0$ in $H_T^{k(q-1)}(\mathrm{pt})$ see [10, 16], and θ_T^k is the image of θ_G^k under the homomorphism $H_{\Sigma_q^{(p)}}^{k(q-1)}(\mathrm{pt}) \to H_G^{k(q-1)}(\mathrm{pt})$, we have $\theta_G^k \neq 0$ for any k. Therefore $\gamma_{M,G} \neq 0$, hence

$$\mathbb{F}_p = H_{\Sigma_q^{(p)}}^{m(q-1)}(M^q, M^q \setminus \Delta) \to H_G^{m(q-1)}(M^q, M^q \setminus \Delta) = \mathbb{F}_p$$

is an isomorphism and the assertion follows. ☐

Lemma 3.7. *Let X be a compact space, or a CW-complex. Let $h\colon X \to M$ be a map such that $h^*\colon H^i(M) \to H^i(X)$ is trivial for any $i > 0$. Then*

1) $h^{q*}\gamma_{M,G} = \theta_G^m$ in $H_G^{m(q-1)}(X^q)$.

2) $\theta_G^m|_{X^q \setminus P} = 0$ where $P = (h^q)^{-1}\Delta(M)$.

Proof. We give the proof in the case when both X and M are compact. Technical details for the proof in general case can be found in [17] where the case $G = T$ was considered.

We use the Nakaoka lemma [13], (see also Lemma 1.1 in [11] or Theorem 2.1 in [9]) to obtain

$$H_G^*(M^q) = H^*(BG, H^*(M)^{\otimes q}) = H^*(G; H^*(M)^{\otimes q}). \tag{3.1}$$

Let us decompose

$$H^*(BG; H^*(M)^{\otimes q}) = H^*(BG) \oplus \mathcal{B}$$

where $H^*(BG) = H^*(BG, H^0(M)^{\otimes q})$, and $\mathcal{B}$ is generated by elements of $H^*(M^q)$ of positive degree. Let us decompose correspondingly $\gamma_{M,G} = \gamma_0 + \gamma_1$. By Lemma 3.6 we obtain

$$\gamma_0 = \theta_G^m \in H^{m(q-1)}(G) = H^{m(q-1)}(BG) = H_G^{m(q-1)}(\mathrm{pt}).$$

The Cartesian power $h^q\colon X^q \to M^q$ induces a zero map in non-equivariant cohomology in positive degrees by the assumption. Then γ_1 is mapped to zero under the map

$$h^{q*}\colon H_G^*(M^q) \to H_G^*(X^q)$$

because the isomorphism (3.1) is functorial and we also have for compact X

$$H_G^*(X^q) = H^*(BG; H^*(X)^{\otimes q}) = H^*(G; H^*(X)^{\otimes q}).$$

Now it follows that $h^{q*}\gamma_1 = 0$ and the first assertion of lemma follows.

Since $\gamma_{M,G}|_{M^q \setminus \Delta} = 0$, we obtain $h^{q*}\gamma_{M,G}|_{X^q \setminus P} = 0$ where $P = (h^q)^{-1}\Delta$. Hence, $\theta_G^m|_{X^q \setminus P} = 0$. ☐

Remark 3.8. Note that the same definition of Haefliger class works also in the case when M is a cohomological manifold (see [2]) over the field $\mathbb{F}_p$ such that the tensor square of its orientation sheaf (over $\mathbb{F}_p$) is constant. It follows that the results of [15] can be extended to the case of maps of T-spaces to nonorientable topological manifolds and to cohomological manifolds under the above condition.

3.6. **Index of G-spaces defined by θ_G.** In this section we consider again a p-subgroup $G \subseteq \Sigma_q^{(p)}$ containing T.

For a G-space X let us introduce its index as follows

$$\operatorname{ind}_{\theta_G} X = \max\{k : \theta_G^k|_X \neq 0\}$$

Recall that $\theta_G^k|_X \in H_G^{k(q-1)}(X^q)$ where the coefficient field $\mathbb{F}_p$ of the cohomology group is suppressed from the notation.

This index possesses usual properties of indicies and we mention some of them needed below.

Lemma 3.9. *1. Monotonicity under equivariant maps of G-spaces: If $X \to Y$ is a G-map, then* $\operatorname{ind}_{\theta_G} X \leq \operatorname{ind}_{\theta_G} Y$.

2. Continuity: Let $A \subset X$ be a closed invariant subspace of a G-space X, then there exists an invariant open neighborhood $U \supset A$ such that $\operatorname{ind}_{\theta_G} A = \operatorname{ind}_{\theta_p} U$.

3. Subadditivity: If $X = A \cup B$ is a union of invariant subspaces then

$$\operatorname{ind}_{\theta_G} X \leq \operatorname{ind}_{\theta_G} A + \operatorname{ind}_{\theta_G} B + 1$$

provided A and B are both open, or A is closed and $B = X \setminus A$.

4. In the case $q = 2$, i.e. for $\Sigma_2 = \mathbb{Z}_2 = T = G_2$, the index $\operatorname{ind}_{\theta_2}$ *coincides with Yang's homological index introduced by Yang in* [18].

Proof. Property 1 follows directly from the definition of the index.

Property 2 is a consequence of the continuity property of cohomology (e.g. the Čech cohomology).

Property 3. Put $\operatorname{ind}_{\theta_G} A = k$ and $\operatorname{ind}_{\theta_G} A = m$. Then $\theta_G^{k+1}|A = 0$ and $\theta_G^{m+1}|B = 0$. If A and B are open it follows from the properties of multiplication of cohomology classes that $\theta_G^{k+1}\theta_G^{m+1}|_{A\cup B} = \theta_G^{k+m+2}|X = 0$, i.e. $\operatorname{ind}_{\theta_G} X \leq k + m + 1$.

The second statement of property 3 follows from the first one and the continuity property 2 of the index.

Property 4. Consider a space X with a free involution $\tau \colon X \to X$. Yang's index of X equals maximal k such that k-th power of the characteristic class of the free involution τ is nontrivial. By definition the characteristic class of τ is the first Stiefel–Whitney class of the linear bundle associated with the covering $\tau \colon X \to X/\tau$ and it is easy to see that this characteristic class of τ coincides with $\theta_2|_X \in H^1_{\mathbb{F}_2}(\mathrm{pt}; \mathbb{Z}_2) = H^1(X/\tau; \mathbb{F}_2)$. □

Theorem 3.10. *Let $\varphi \colon Y \to E$ be an equivariant map of G-spaces and $P \subset E$ be an G-invariant closed subspace. If* $\operatorname{ind}_{\theta_G} Y = n > m = \operatorname{ind}_{\theta_G}(E \setminus P)$ *then $\varphi^{-1}(P) \neq \emptyset$ and* $\operatorname{ind}_{\theta_G}\varphi^{-1}(P) \geq n - m$.

Proof. From property 1 of the index we have $\operatorname{ind}_{\theta_G}(Y \setminus \varphi^{-1}(P)) \leq \operatorname{ind}_{\theta_G}(E \setminus P) = m$. This inequality shows that $\varphi^{-1}(P)$ cannot be empty.

Arguing by contradiction assume that $\operatorname{ind}_{\theta_G} \varphi^{-1}P < n - m - 1$. Then from property 3 we obtain

$$\operatorname{ind}_{\theta_G} Y \leq \operatorname{ind}_{\theta_G} \varphi^{-1}P + \operatorname{ind}_{\theta_G}(Y \setminus \varphi^{-1}(P)) + 1 < n - m - 1 + m + 1 = n,$$

so $\operatorname{ind}_{\theta_G} Y < n$ contradicting with the assumption. □

Definition 3.11. Suppose Y is a G-space. For a G-map $\varphi \colon Y \to (X')^q$ we put

$$C(\varphi) = \varphi^{-1}(\Delta)$$

where

$$\Delta = \Delta(X') = \{(x', \dots, x') \in (X')^q \ : \ x' \in Y'\}$$

is the diagonal in $(X')^q$.

Definition 3.12. Let X be a metric space, Y be a G-invariant subspace of X^q, and $f : X \to X'$ be a continuous map to a topological space X'. Put

$$B(f) = \{(x_1, \dots, x_q) \in Y \subset X^q \ : \ f(x_1) = \dots = f(x_q)\}.$$

Obviously, $B(f) = C(\varphi)$ for $\varphi = f^q \colon X^q \to (X')^q$.

Remark 3.13. We are mainly interested in the case $Y \subset K^q(X)$, where $K^q(X)$ is a configuration spaced based on a space X. However for $G = T$ the case when $Y \subset X^q$ is an invariant subspace such that T-action on Y has no fixed points is also interesting. Note that for a T-space X there exists an equivariant embedding $X \to X^q$ (see [17]). For example, for the space X with an action of the cyclic group $\mathbb{Z}_p$ with generator τ this is a map $x \to (x, \tau x, \dots, \tau^{p-1}x) \in X^p$. In particular for the space X with involution τ we have the equivariant embedding $x \to (x, \tau x) \in X^2$.

Since (from the equivariant Thom isomorphism and annihilation of the Euler class)

$$\operatorname{ind}_{\theta_G} ((\mathbb{R}^m)^q \setminus \Delta) = \operatorname{ind}_{\theta_G} ((\alpha_q)^m \setminus \{0\}) = m - 1,$$

we obtain:

Corollary 3.14. *Let Y be a G-space and $\varphi \colon Y \to (\mathbb{R}^m)^q$ be a G-map. If $\operatorname{ind}_{\theta_G} Y = n \geq m$ then $C(\varphi) \neq \emptyset$ and $\operatorname{ind}_\theta C(\varphi) \geq n - m$.*

In particular, if $Y \subset X^q$ is an invariant subspace and $f \colon X \to \mathbb{R}^m$ then $B(f) \neq \emptyset$ and $\operatorname{ind}_\theta B(f) \geq n - m$.

Remark 3.15. The classical Bourgin–Yang theorem for $\mathbb{Z}_2$-spaces and maps to Euclidean spaces follows easily from the above result.

Let Y be a G-space and $\varphi \colon Y \to X^q$ a G-map. Let $h \colon X \to M$ be a map. Then $\psi := h^q \circ \varphi \colon Y \to M^q$ is an equivariant map and

$$C(\psi) = \varphi^{-1} \circ (h^q)^{-1}(\Delta(M)).$$

Theorem 3.16. *Let Y be a G-space, $\varphi \colon Y \to X^q$ a G-map and $h \colon X \to M$ be a continuous map to an m-dimensional topological manifold of a space X which is compact or a CW.*

Assume that $h^* \colon H^i(M) \to H^i(X)$ is trivial in dimensions $i > 0$. If $\operatorname{ind}_{\theta_G} Y = n \geq m$ then $\operatorname{ind}_{\theta_G} C(\psi) \geq n - m$. In particular, $C(\psi) \neq \emptyset$.

Proof. Now we can use theorem 3.10, however it is easier to repeat the argument. From lemma 3.7 and monotonicity property of the index we obtain $\operatorname{ind}_{\theta_G}(Y \setminus C(\psi)) \leq m - 1$. Thus $C(\psi)$ cannot be empty.

From Property 3 of the index we obtain

$$\operatorname{ind}_{\theta_G} Y \leq \operatorname{ind}_{\theta_G} C(\psi) + \operatorname{ind}_{\theta_G}(Y \setminus C(\psi)) + 1 \leq \operatorname{ind}_{\theta_G} C(\psi) + m.$$

So, if $\operatorname{ind}_{\theta_G} C(\varphi) < n - m$ then $\operatorname{ind}_{\theta_G} Y < n$, contradicting with our assumptions. □

4. Proof of Theorem 2.4

We apply the above results for the p-Sylow group $G = \Sigma_q^{(p)}$.

Using the index notation we have $\operatorname{ind}_{\theta_p} K^q(\mathbb{R}^d) = d - 1$, and Lemma 3.5 can be stated as follows: $\operatorname{ind}_{\theta_p} Z_f \geq \operatorname{ind}_{\theta_p} X - 1$.

For $y \in K^q(L) = K^q(\mathbb{R}^{n+1})$ we have a partition $(V_1(y), \dots, V_q(y))$ of X and a $\Sigma_q^{(p)}$-map $\varphi_1 : K^q(\mathbb{R}^{n+1}) \to \mathbb{R}^q$ defined as $\varphi_1(y) = (\mu_1(V_1(y)), \dots, \mu_1(V_q(y))) \in \mathbb{R}^q$.

Put $Y_1 = C(\varphi_1)$ and from corollary 3.14 we obtain $\operatorname{ind}_{\theta_p} Y_1 \geq n - 1$. Note also that for any $y \in Y_1$ the partition $V_1(y), \dots, V_q(y)$ consists of sets with nonempty interiors, thus justifying the Remark 2.5.

Define $\varphi_2 \colon Y_1 \to (\mathbb{R}^{n-m-1})^q$ as $\varphi_2(y) = (h(V_1(y)), \dots, h(V_q(y)))$ where $h = (\mu_2, \dots, \mu_{n-m})$. Applying corollary 3.14 again we see that $\operatorname{ind}_{\theta_p} C(\varphi_2) \geq m$ and we put $Y_2 = C(\varphi_2)$.

Finally we define equivariant map $\varphi_3 \colon Y_2 \to X^q$ as $\varphi_3(y) = (c(V_1(y)), \dots, c(V_q(y)))$. Applying theorem 3.16 to maps φ_3 and $f : X \to M$ we finish the proof.

Remark 4.1. Alternatively we can apply theorem 3.16 to maps

$$\psi : Y_1 \to X_1^q = (X \times \mathbb{R}^{n-m-1})^q \quad \text{and} \quad f_1 \colon X_1 \to M_1 = M \times \mathbb{R}^{n-m-1}$$

defined as $\psi(y) = (r(V_1(y)), \dots, r(V_q(y))$ where $r(V_i(y)) = (c(V_i(y)), h(V_i(y)))$ and $f_1 = (f, \mathrm{id}_{\mathbb{R}^{n-m-1}})$

5. Proof of Theorem 1.1

First note that for $n = m$ the theorem follows from the ordinary Borsuk–Ulam theorem for maps to manifolds (see [3] of [17] for example). In this case some two antipodal points $x, -x \in S^n$ are mapped to a single point in M, and the set $\{x, -x\}$ is itself a standard 0-sphere.

In case $n > m$ the proof follows literally the proof in [12]. The only thing we have to check is whether Theorem 4 of [12] can be generalized for maps to M, that is we have to prove the following:

Lemma 5.1. *Suppose $h : S^n \to M$ is a continuous map satisfying assumptions of Theorem 1.1. Then for any $q = 2^l$ the sphere can be partitioned into q convex parts $V_1^q, V_2^q, \dots, V_q^q$ so that*

1) the measures $\mu V_1^q, \mu V_2^q, \dots, \mu V_q^q$ are equal;

2) the mass centers $c(V_1^q), c(V_2^q), \dots, c(V_q^q)$ are mapped by h to the same point;

3) for any $\varepsilon > 0$ there exists N such that for any $q = 2^l > N$ and any $1 \leq j \leq q$ the set V_j^q is ε-close to some m-dimensional subsphere of S^n (i.e. intersection $S^n \cap V$ with an $(m+1)$-dimensional linear subspace $V \subset \mathbb{R}^{n+1}$).

Proof. Following [12] we reproduce the proof of Theorem 2.4 in a modified form. Consider some linear space L of homogeneous linear functions on $\mathbb{R}^{n+1} \supset S^n$. The corresponding partitions will be partitions into convex sets.

Let us restrict the symmetry group to $\Sigma_q^{(2)}$ and pass from the configuration space $K^q(L)$ to a certain $\Sigma_q^{(2)}$-invariant subspace $Q^q(L)$, this space was defined explicitly in [6] to study the cohomology of configuration spaces and used in [12] to prove the sphere waist theorem.

Definition 5.2. Let $Q^q(L)$ be defined inductively as follows. Take some small $\delta > 0$, and let $Q^q(L)$ contain the configurations of q points with following conditions:

1) for $q = 1$ the space $Q^q(L)$ contains only one configuration, where one point is at the origin;

2) for $q \geq 2$ the first $q/2$ points form a configuration from $Q^{q/2}(L)$ scaled by δ and shifted by a vector v of length 1;

3) for $q \geq 2$ the last $q/2$ points form another configuration from $Q^{q/2}(L)$ scaled by δ and shifted by $-v$.

Topologically this space is a product of $q-1$ spheres $S^{\dim L - 1}$, corresponding to different translation vectors v on the stages of its construction.

If δ tends to zero, the subspace $Q^q(L)$ corresponds to binary partitions of S^n by hyperplanes through the origin in $\mathbb{R}^{n+1}$ orthogonal to L and arranged in a full binary tree of height l (note $2^l = q$). The main result of [6] shows that the natural map

$$H^*_{\Sigma_q^{(2)}}(K^q(L), \mathbb{F}_2) \to H^*_{\Sigma_q^{(2)}}(Q^q(L), \mathbb{F}_2)$$

is an injection, therefore Lemma 3.4 is also valid for $Q^q(L)$, i.e.

$$e(\alpha_q)^{\dim L-1} \neq 0 \in H^*_{\Sigma_q^{(2)}}(Q^q(L), \mathbb{F}_2).$$

Note that unlike Theorem 2.4 we have to equipartition only one measure, so we may take as L any linear subspace of homogeneous linear functions of dimension $m+2$. Moreover, since the space $Q^q(L)$ has hierarchical structure, we may replace L by a different $(m+2)$-dimensional L_i on each level $1 \leq i \leq l-1$ of the binary tree. Denote the corresponding configuration space by $Q^q(L_1, L_2, \ldots, L_{l-1})$. This space is $\Sigma_q^{(2)}$-equivariantly homotopy equivalent to $Q^q(\mathbb{R}^{m+2})$, so the Euler class $e(\alpha_q)^{m+1}$ is still nonzero in its $\Sigma_q^{(2)}$-equivariant cohomology (with $\mathbb{F}_2$ coefficients). In [12] it was shown that by selecting L_i to be uniformly distributed in some sense (for large enough l), we obtain Claim 3 of this Lemma. Claims 1 and 2 are obtained as in the proof of Theorem 2.4; partitioning one measure "takes" $e(\alpha_q)$ and the coincidence in M "takes" the remaining $e(\alpha_q)^m$. □

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